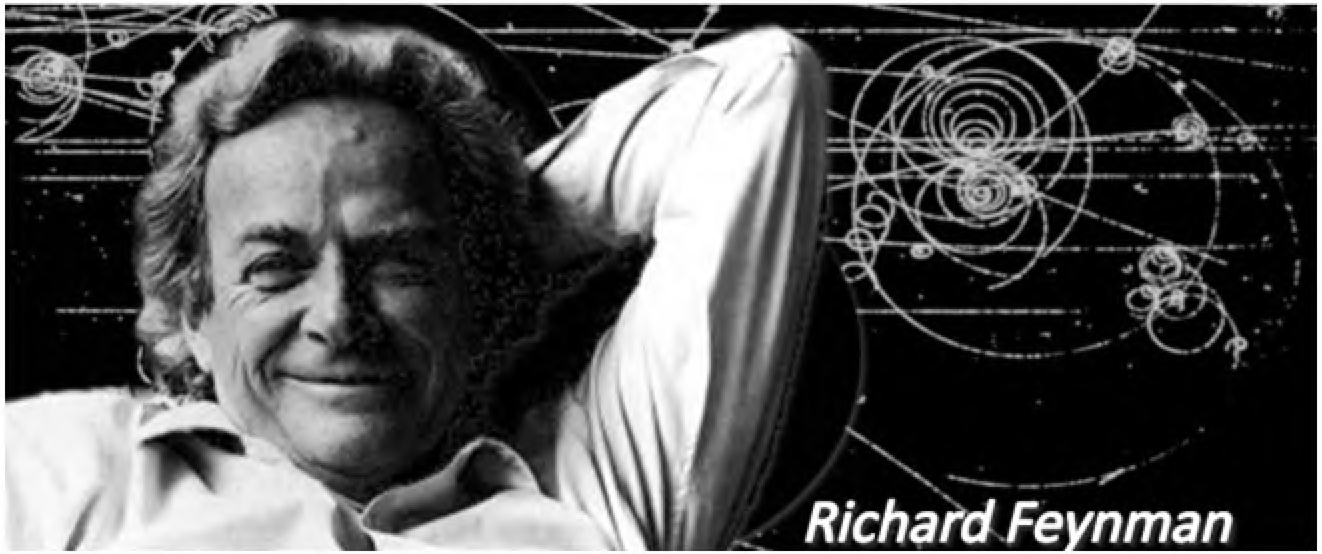
Electromagnetism I

Faculty of Physics-Kharazmi University

Dr. Faramarz Kanjouri

Spring 2023

دانشگاه خوارزمی



اگر همواره مانند گذشته بیندیشید، همیشه همان چیزهایی را به‌دست می‌آورید که تا کنون کسب کرده‌اید

فاینمن

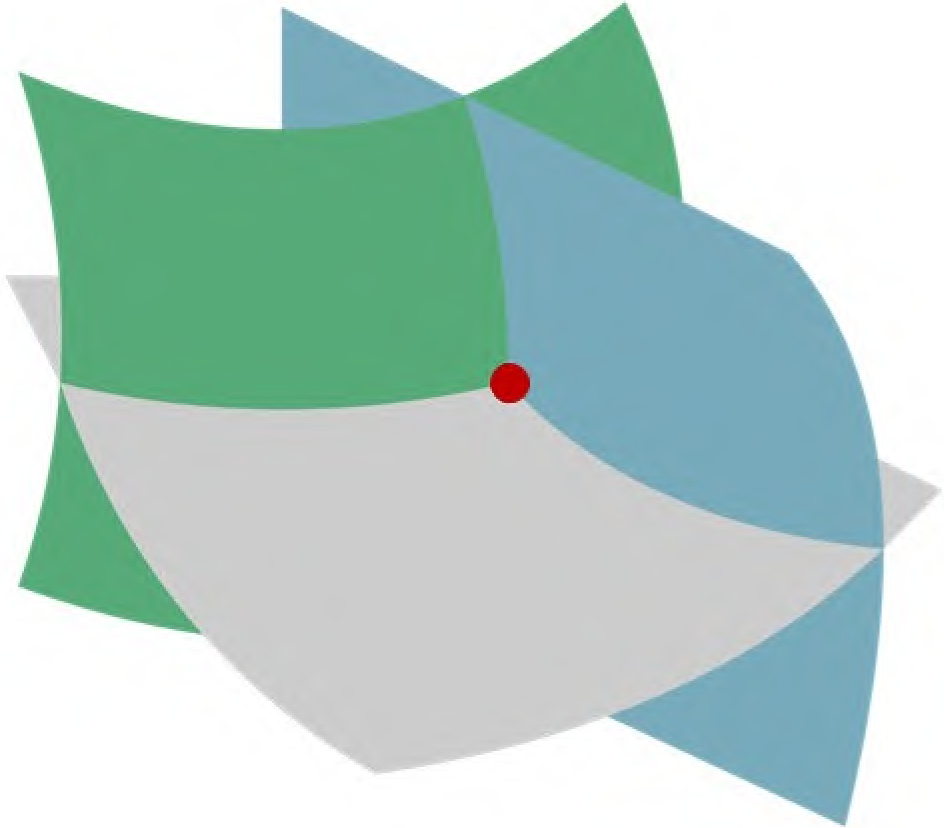
دانشگاه خوارزمی

درس سوم

دستگاه‌های مختصات

Coordinate Systems

دانشگاه خوارزمی

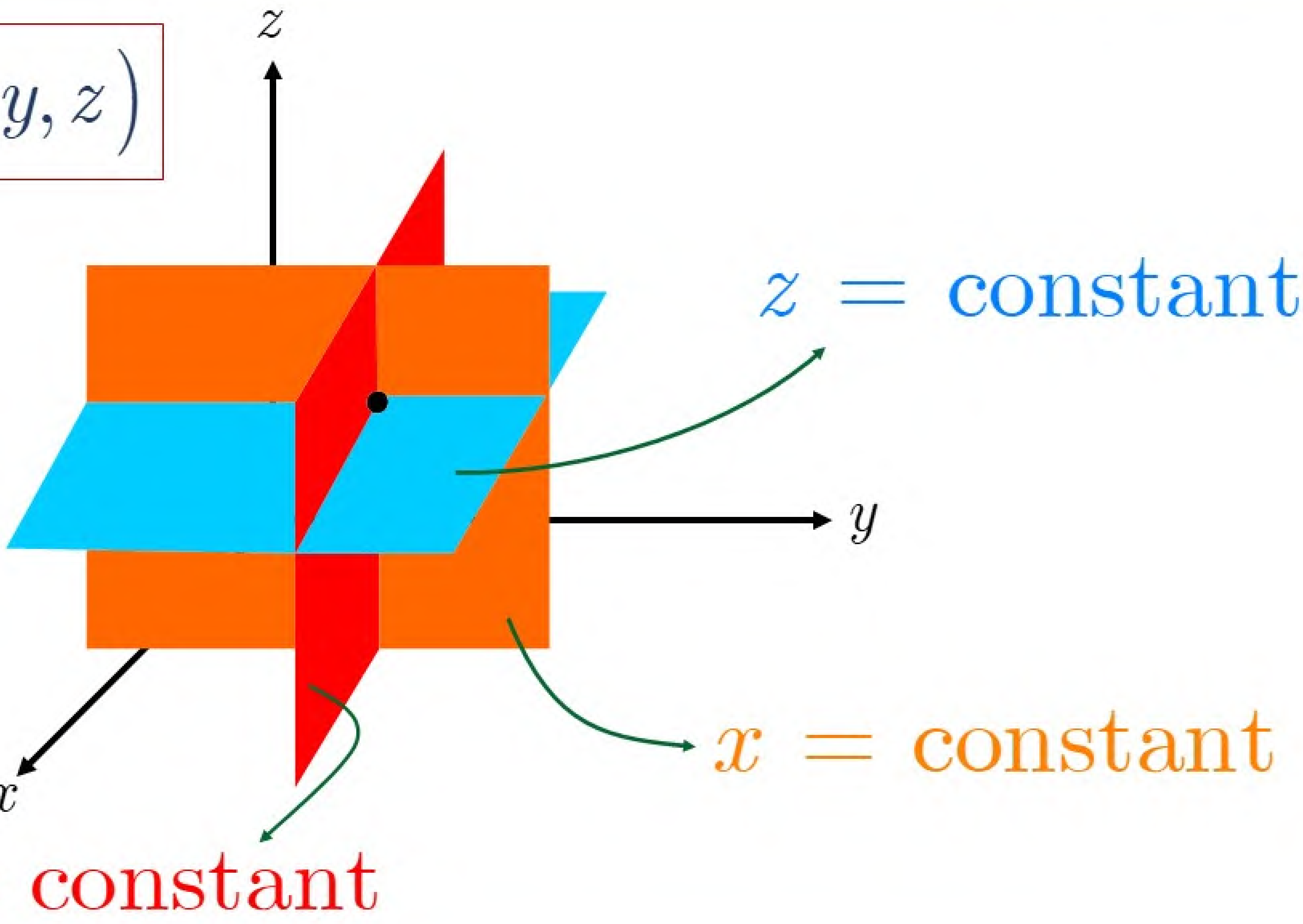


$u_1 = \text{constant}$

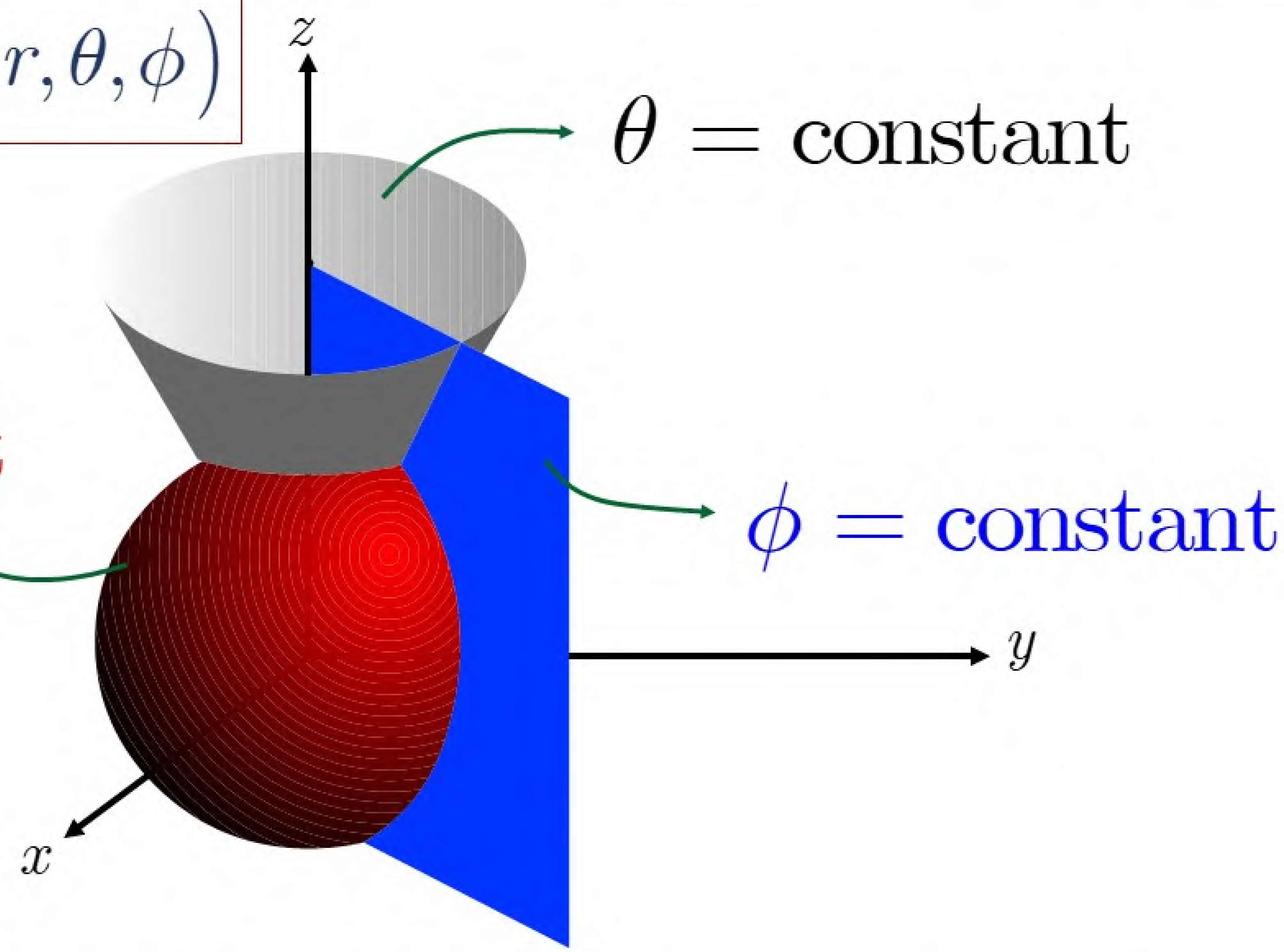
$u_2 = \text{constant}$

$u_3 = \text{constant}$

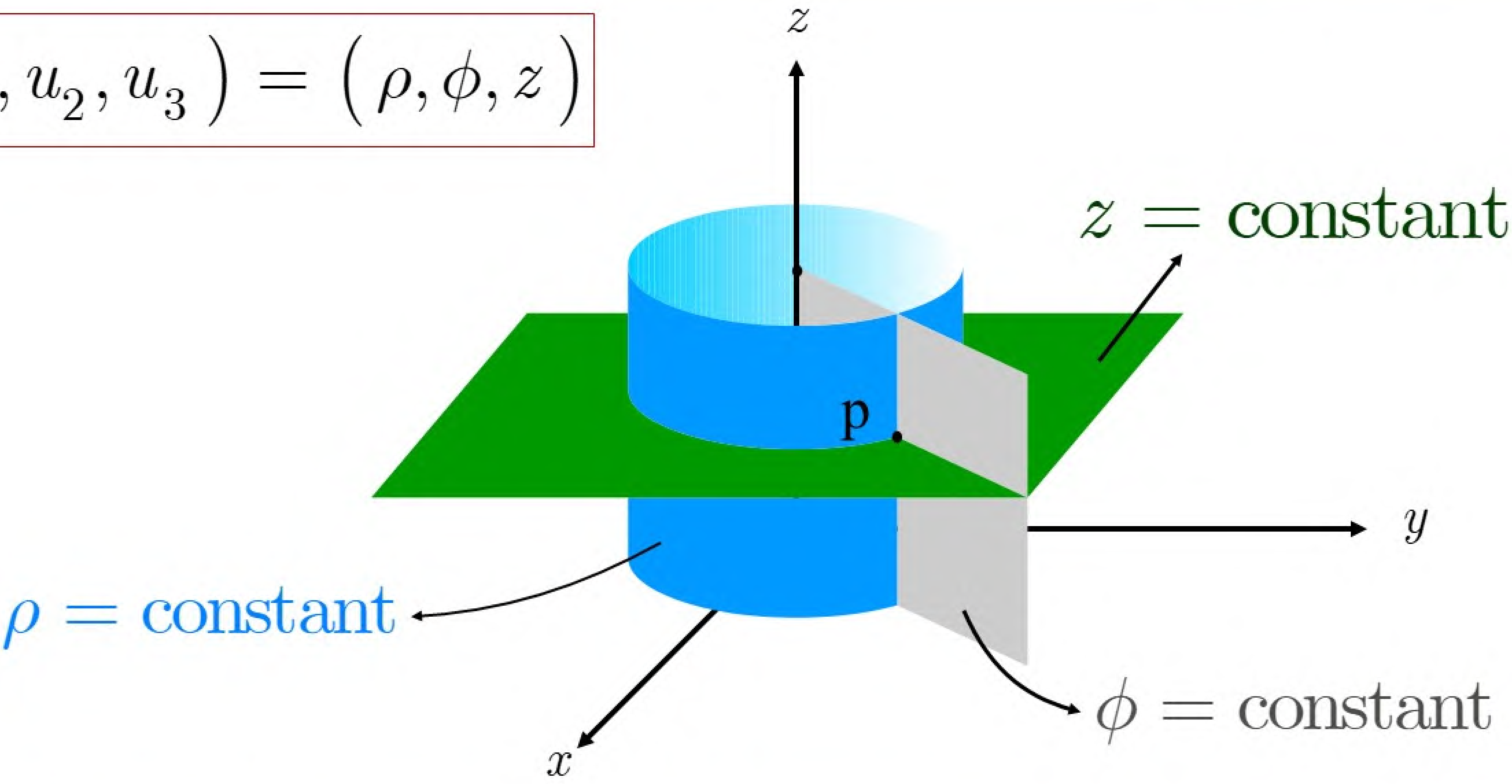
$$\boxed{\left(u_1, u_2, u_3\right) = \left(x, y, z\right)}$$

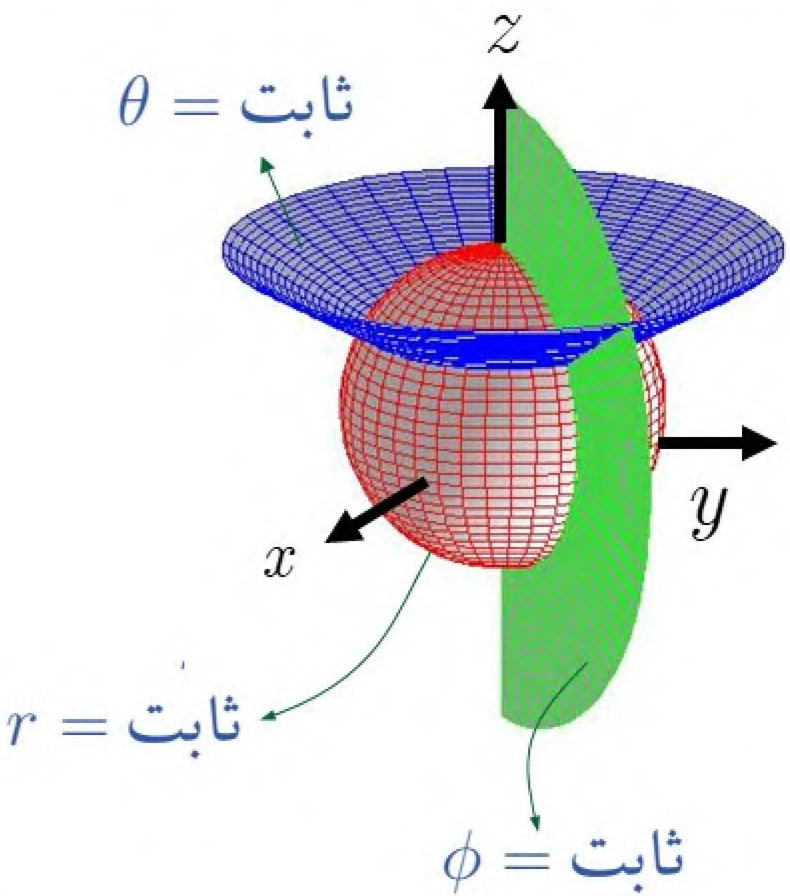


$$\left(u_1, u_2, u_3\right) = \left(r, \theta, \phi\right)$$

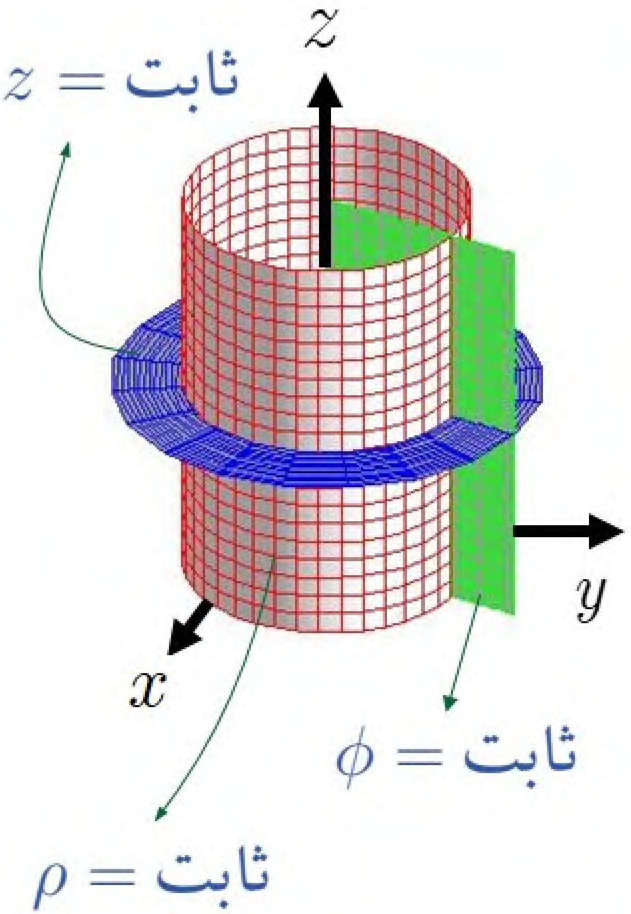


$$\boxed{\left(u_1, u_2, u_3 \right) = \left(\rho, \phi, z \right)}$$

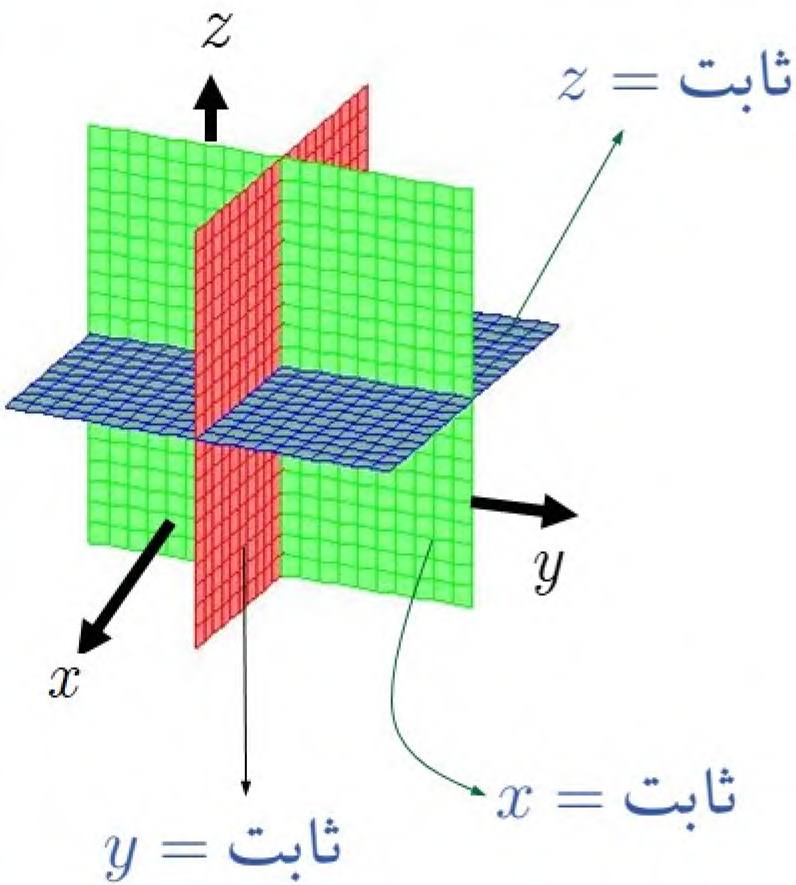




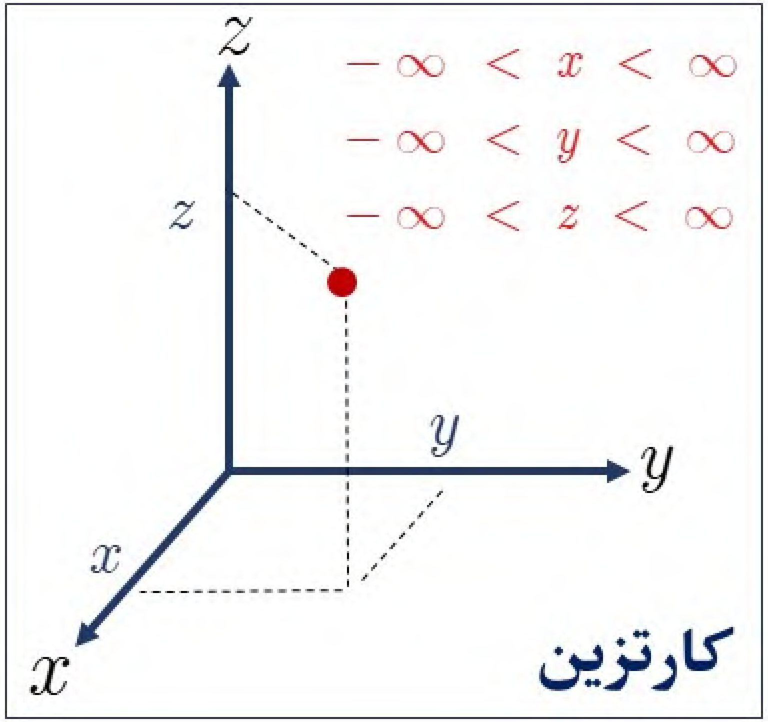
$$\left(u_1, u_2, u_3\right) = \left(r, \theta, \phi\right)$$

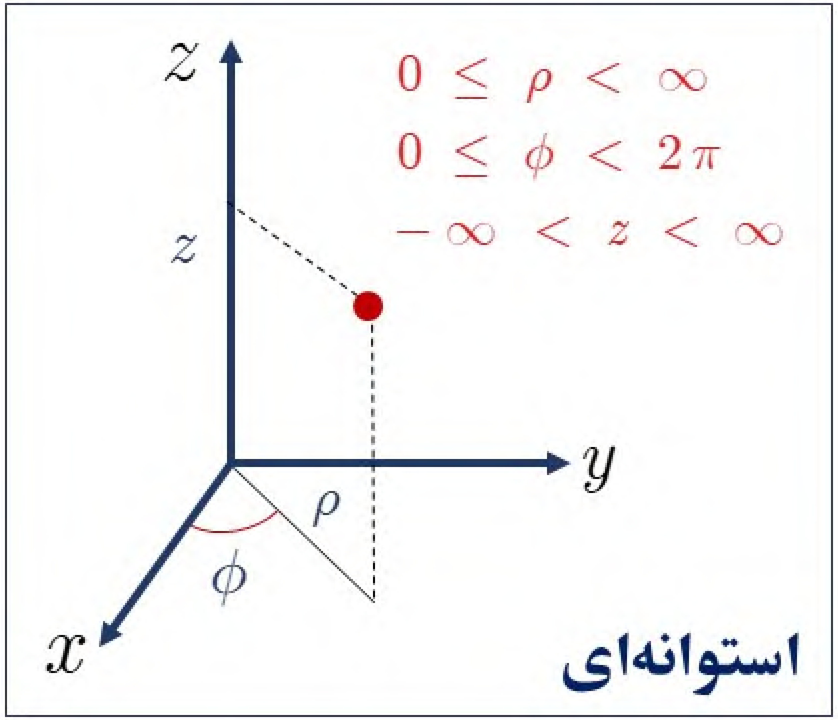


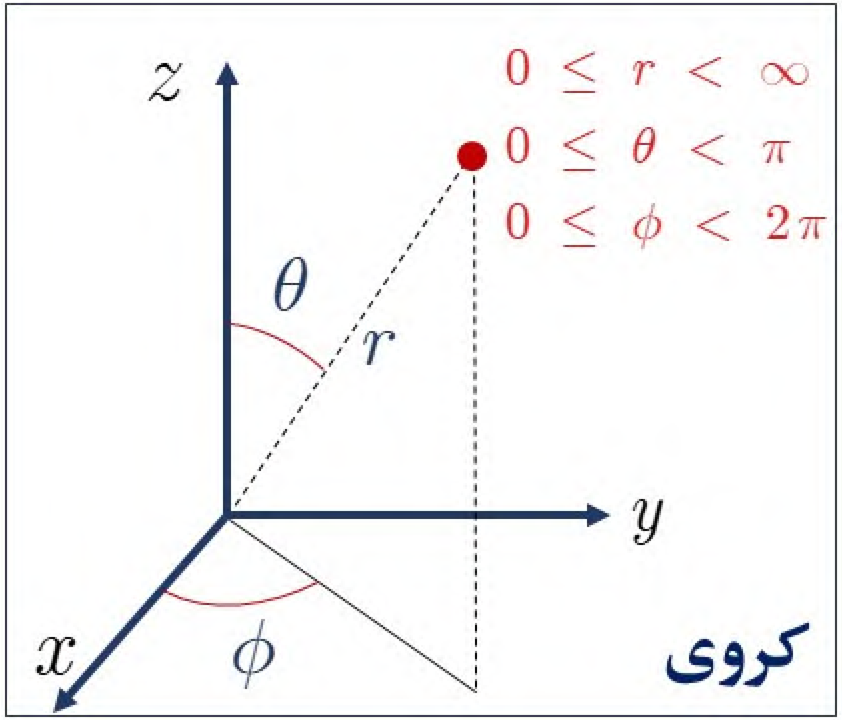
$$\left(u_1, u_2, u_3\right) = \left(\rho, \phi, z\right)$$



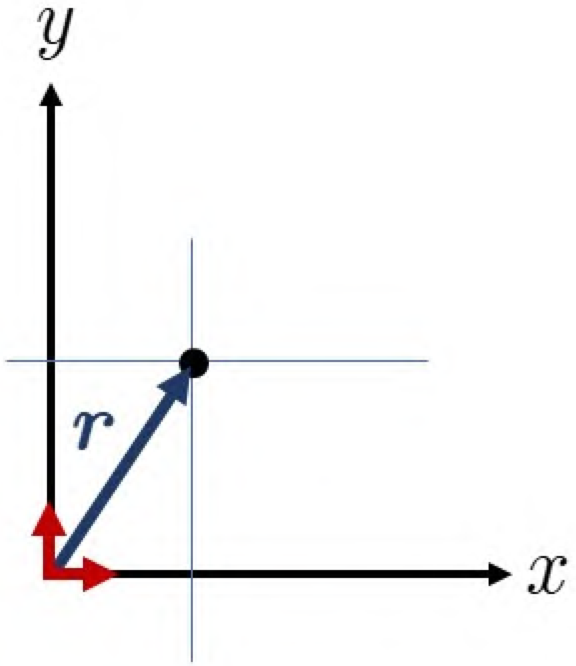
$$\left(u_1, u_2, u_3\right) = \left(x, y, z\right)$$


$-\infty < x < \infty$
$-\infty < y < \infty$
$-\infty < z < \infty$
x y z
کارتزین

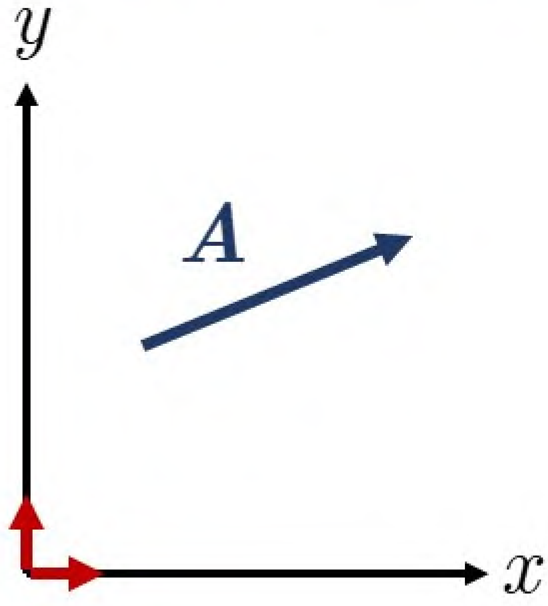

$0 \leq \rho < \infty$
$0 \leq \phi < 2\pi$
$-\infty < z < \infty$
x y z ρ ϕ
استوانه‌ای


$0 \leq r < \infty$
$0 \leq \theta < \pi$
$0 \leq \phi < 2\pi$
x y z r θ ϕ
کروی

برای درک بهتر مسئله، حالت دو بعدی را بررسی می کنیم



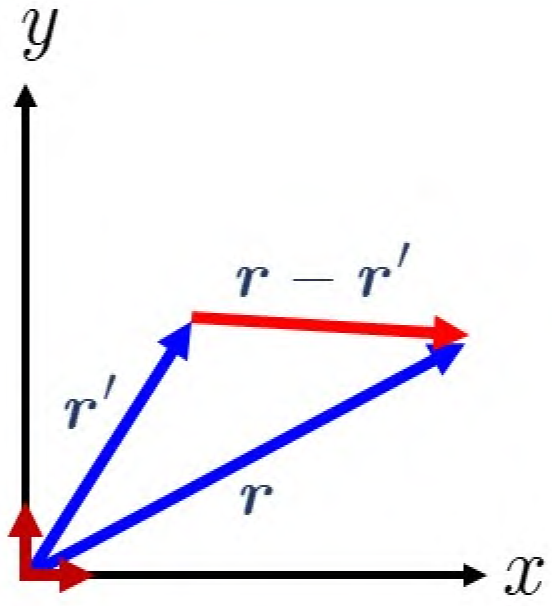
$$\boldsymbol{r} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y$$



$$\boldsymbol{A} = A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y$$

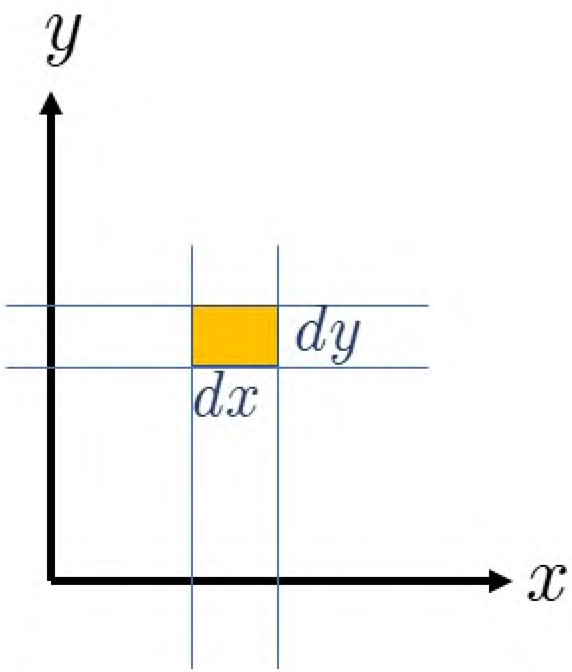
$$A_x = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_x$$

$$A_y = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_y$$



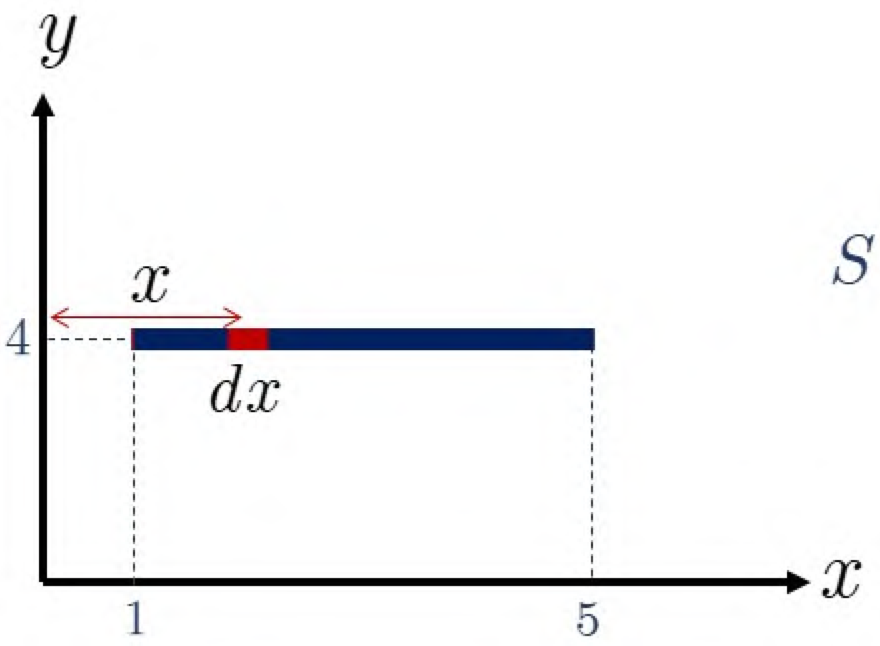
$$\boldsymbol{r}-\boldsymbol{r}' = (x-x')\hat{\boldsymbol{e}}_x + (y-y')\hat{\boldsymbol{e}}_y$$

$$d = \left|\boldsymbol{r}-\boldsymbol{r}'\right| = \sqrt{(x-x')^2+(y-y')^2}$$



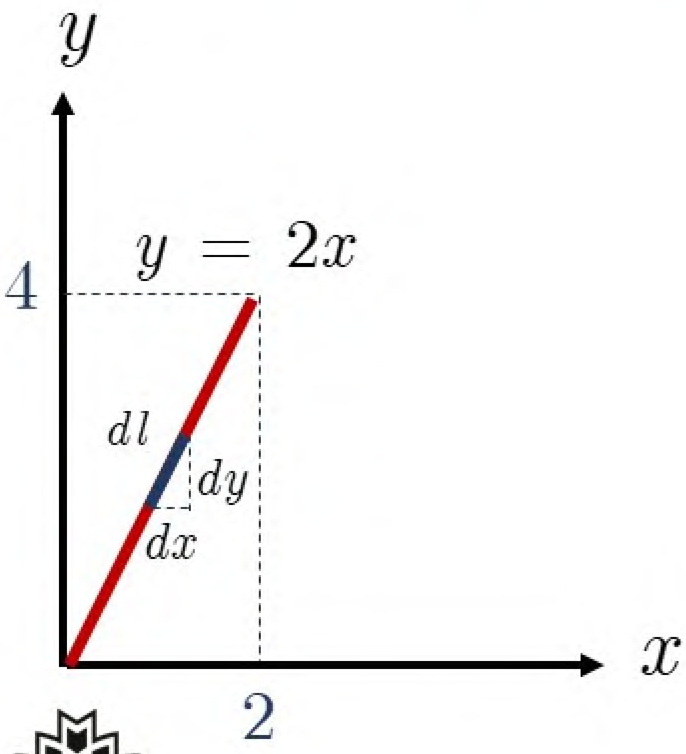
$$d\boldsymbol{l} \equiv d\boldsymbol{r} = dx\hat{\boldsymbol{e}}_x + dy\hat{\boldsymbol{e}}_y$$

$$da = dxdy$$



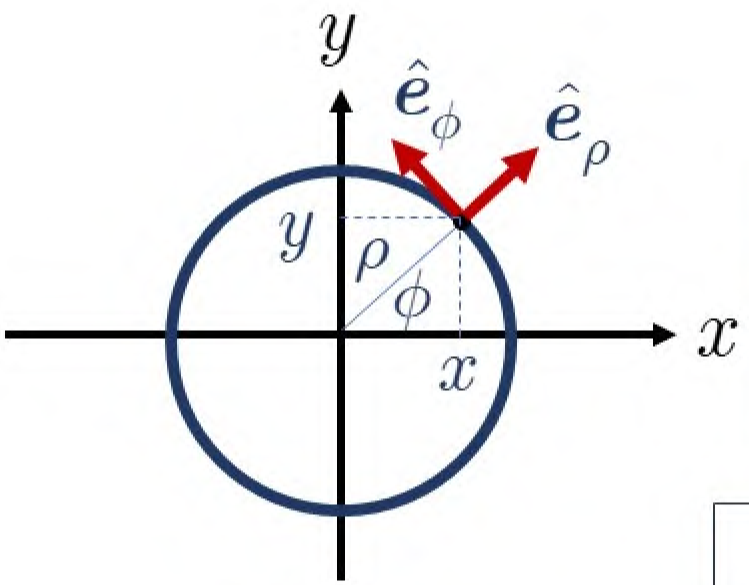
$$S = \int_1^5 dx = x\Big|_1^5 = 5 - 1 = 4$$

$$L = \int_{x=0,y=0}^{x=2,y=4} dl = \int_{x=0,y=0}^{x=2,y=4} \sqrt{dx^2 + dy^2} = \int_{x=0,y=0}^{x=2,y=4} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$



$$L = \int_{x=0}^{x=2} dx\sqrt{1 + (2)^2}$$

$$L = \sqrt{5}\int_{x=0}^{x=2} dx = 2\sqrt{5}$$

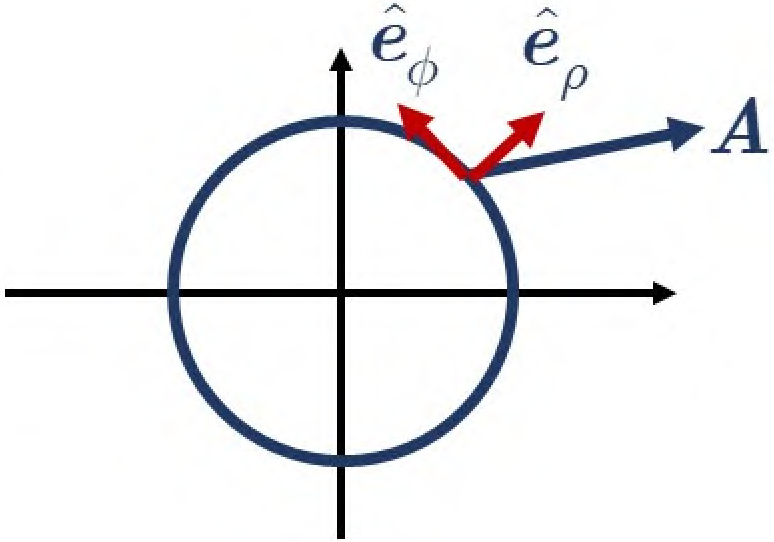


$$\boldsymbol{r} = \rho \hat{\boldsymbol{e}}_\rho$$

$$\boxed{\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \end{aligned}}$$

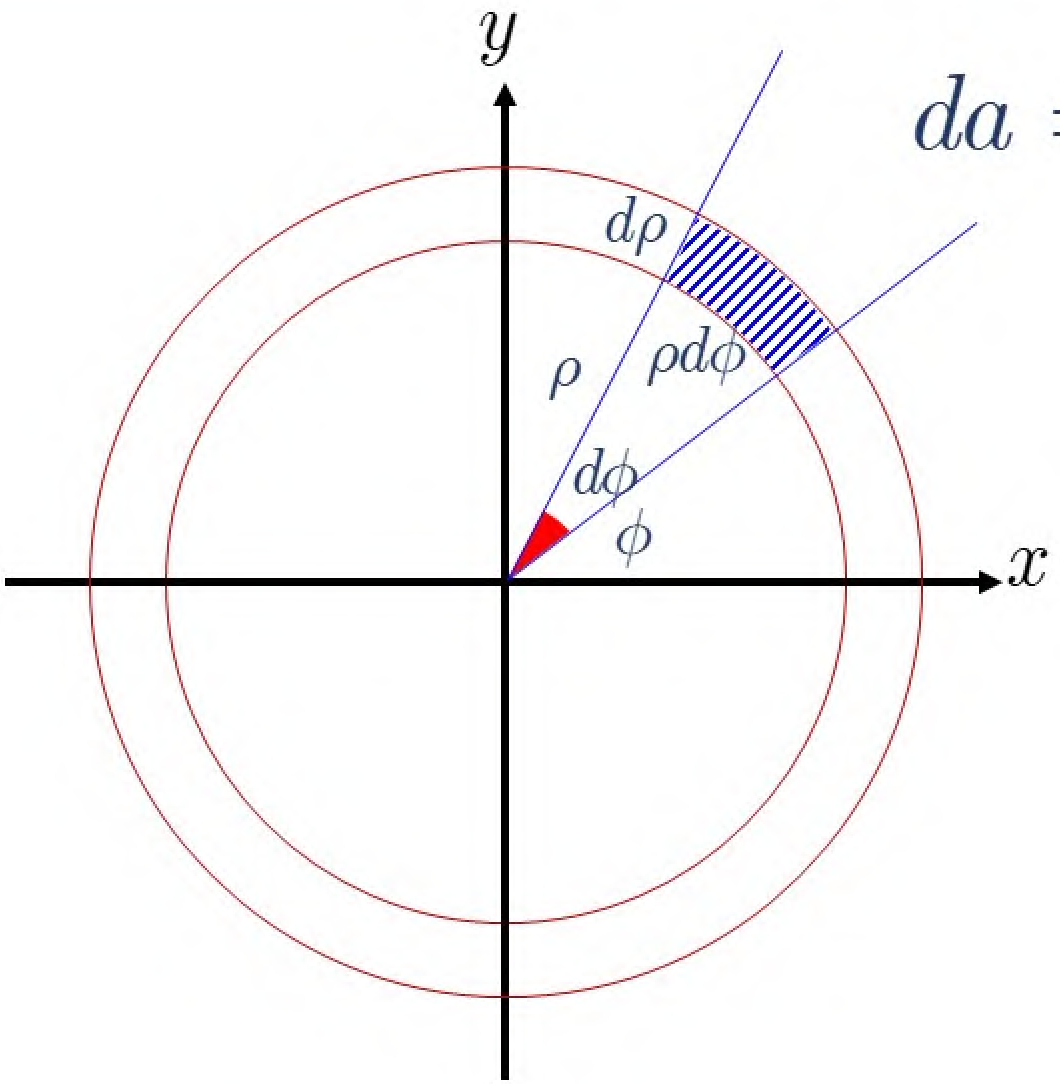
$$\boxed{\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1} \frac{y}{x} \end{aligned}}$$

$$\boxed{\begin{aligned} \cos\phi &= \frac{x}{\sqrt{x^2 + y^2}} \\ \sin\phi &= \frac{y}{\sqrt{x^2 + y^2}} \end{aligned}}$$

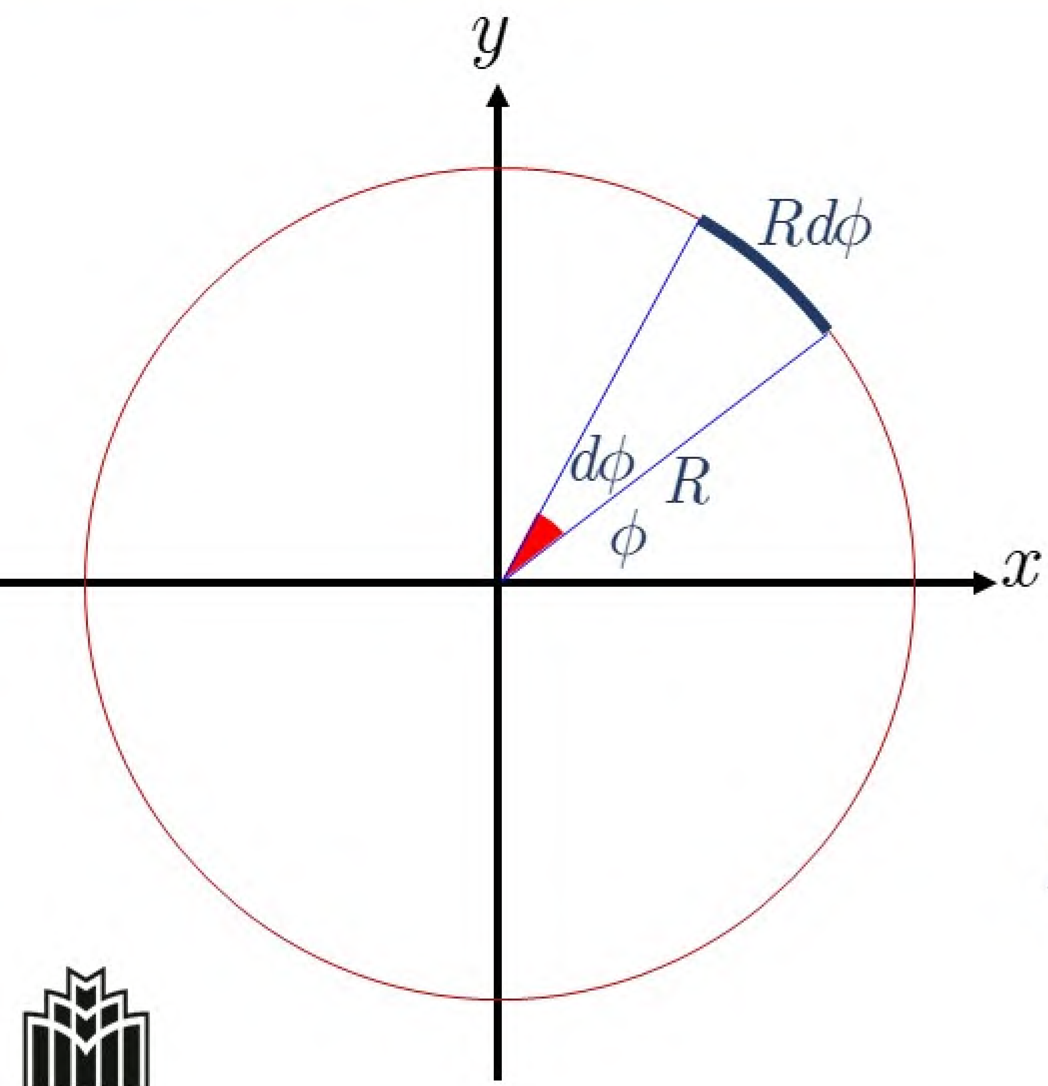


$$\boldsymbol{A} = A_\rho \hat{\boldsymbol{e}}_\rho + A_\phi \hat{\boldsymbol{e}}_\phi$$

عنصر سطح در دستگاه مختصات قطبی (دو بعدی)

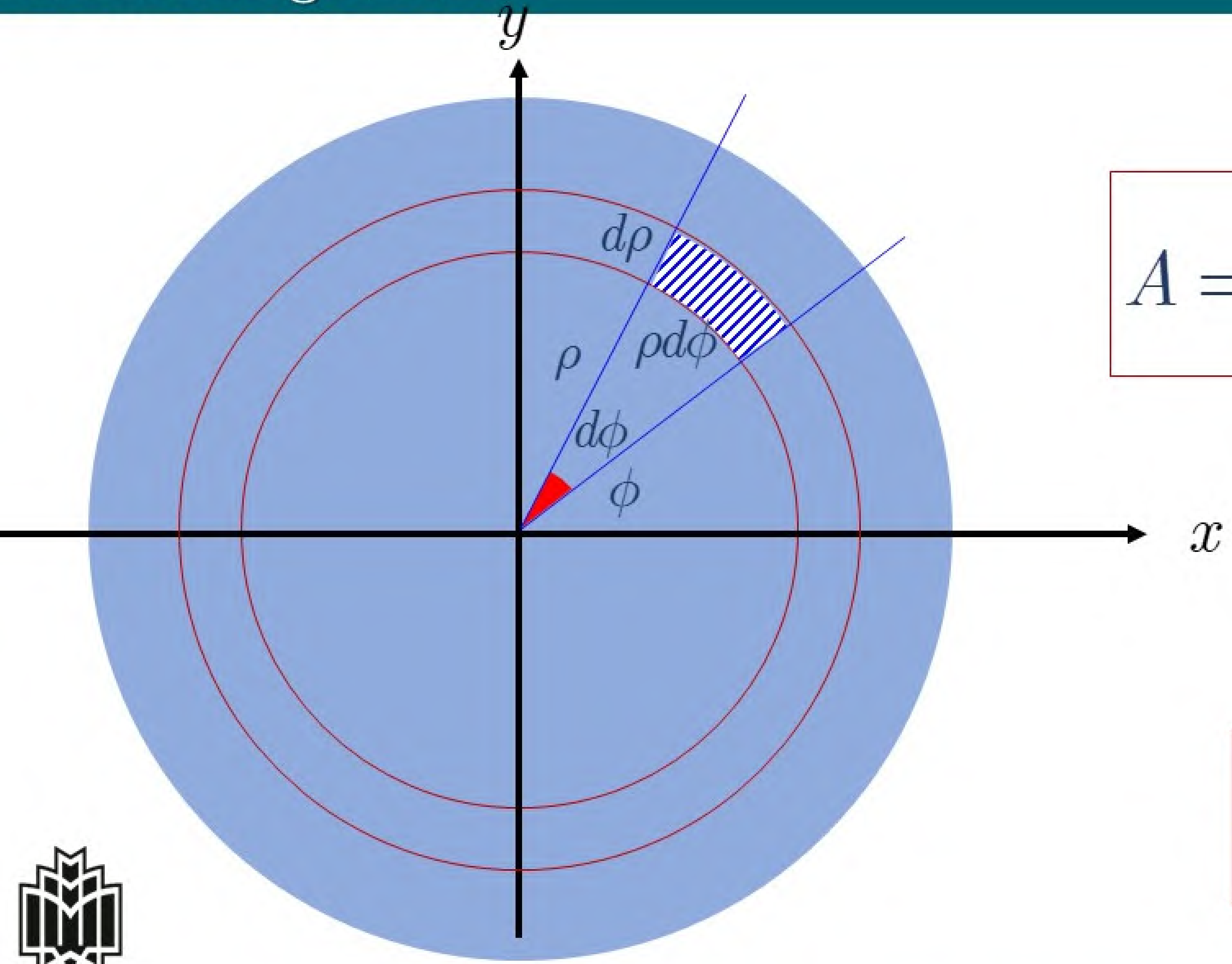


$$da = \rho d\rho d\phi$$



$$dl = Rd\phi$$

$$L = \int dl = R\int_0^{2\pi} d\phi = R\phi\Big|_0^{2\pi} = 2\pi R$$

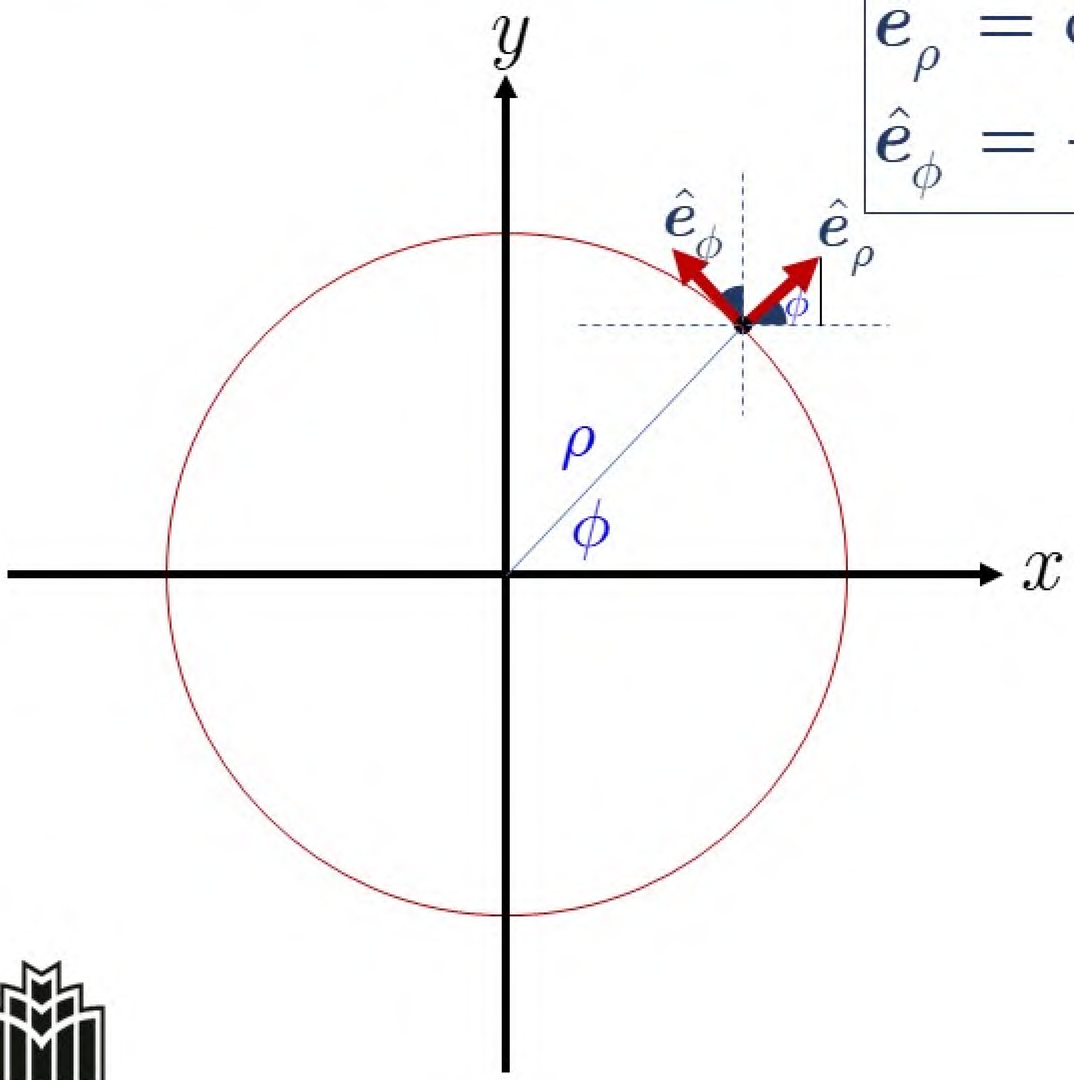


$$da = \rho d\rho d\phi$$

$$\boxed{A = \int da = \int_0^{2\pi} \int_0^R \rho d\rho d\phi}$$

$$\boxed{A = \int_0^{2\pi} d\phi \left[\int_0^R \rho d\rho \right]}$$

$$\boxed{A = \left(2\pi\right)\left(\frac{1}{2}R^2\right) = \pi R^2}$$



$$\hat{\boldsymbol{e}}_\rho = \cos\phi\hat{\boldsymbol{e}}_x + \sin\phi\hat{\boldsymbol{e}}_y$$
$$\hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{e}}_x + \cos\phi\hat{\boldsymbol{e}}_y$$

$$\hat{\boldsymbol{e}}_x = \cos\phi\hat{\boldsymbol{e}}_\rho - \sin\phi\hat{\boldsymbol{e}}_\phi$$
$$\hat{\boldsymbol{e}}_y = \sin\phi\hat{\boldsymbol{e}}_\rho + \cos\phi\hat{\boldsymbol{e}}_\phi$$

$$A_x = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_x = A_\rho\hat{\boldsymbol{e}}_\rho\cdot\hat{\boldsymbol{e}}_x + A_\phi\hat{\boldsymbol{e}}_\phi\cdot\hat{\boldsymbol{e}}_x$$

$$A_y = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_y = A_\rho\hat{\boldsymbol{e}}_\rho\cdot\hat{\boldsymbol{e}}_y + A_\phi\hat{\boldsymbol{e}}_\phi\cdot\hat{\boldsymbol{e}}_y$$

$$\boxed{\begin{aligned} A_x &= A_\rho\cos\phi - A_\phi\sin\phi \\ A_y &= A_\rho\sin\phi + A_\phi\cos\phi \end{aligned}}$$

دانشگاه خوارزمی

$$\boxed{\begin{aligned} A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\ A_y &= A_\rho \sin\phi + A_\phi \cos\phi \end{aligned}}$$

$$\boxed{\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} A_\rho \\ A_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & -\dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \end{pmatrix}}$$

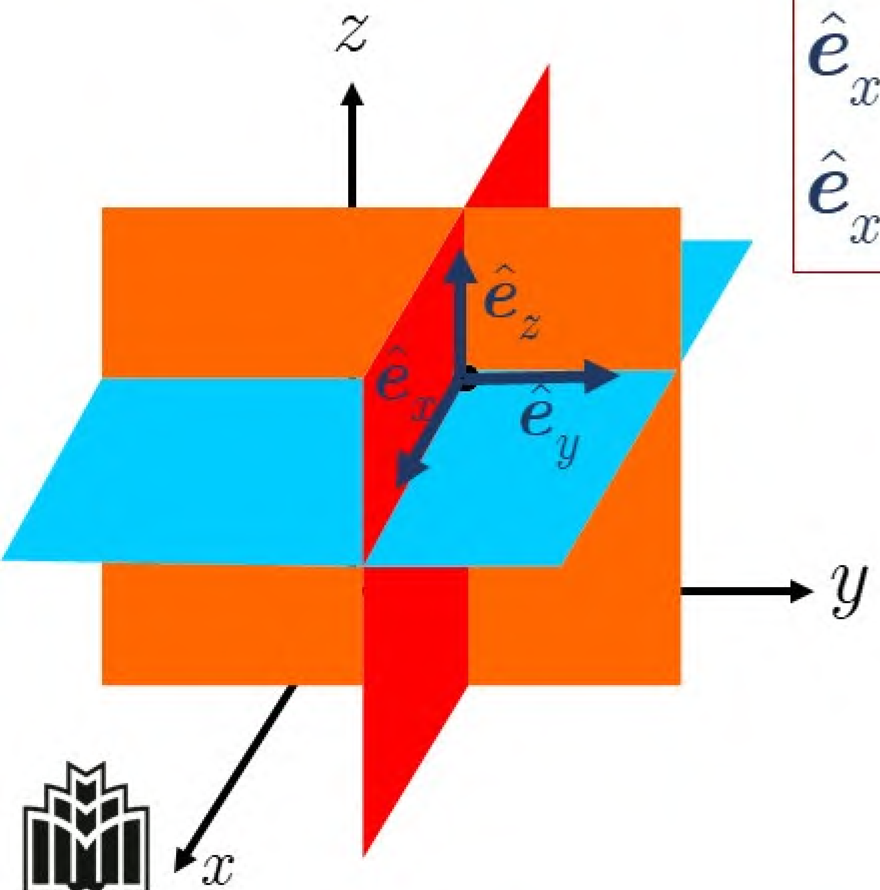
دانشگاه خوارزمی

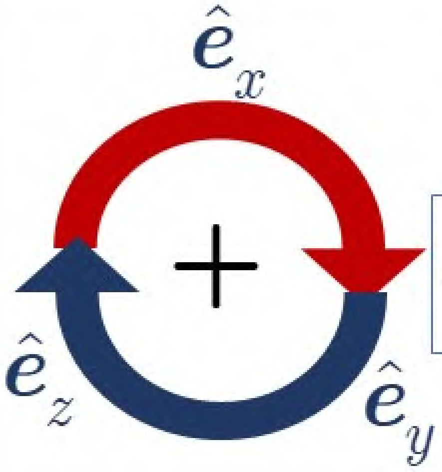
$$\boldsymbol{r} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$$

$$\boldsymbol{A} = A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z$$

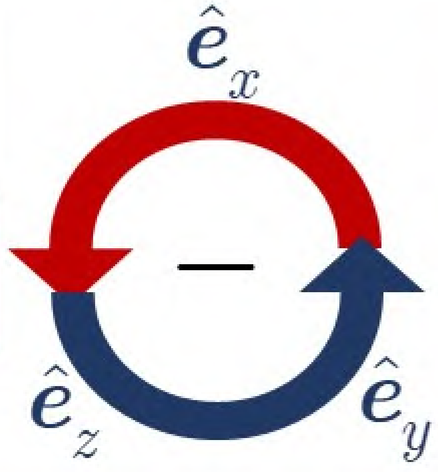
$$A_x = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_x$$
$$A_y = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_y$$
$$A_z = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_z$$

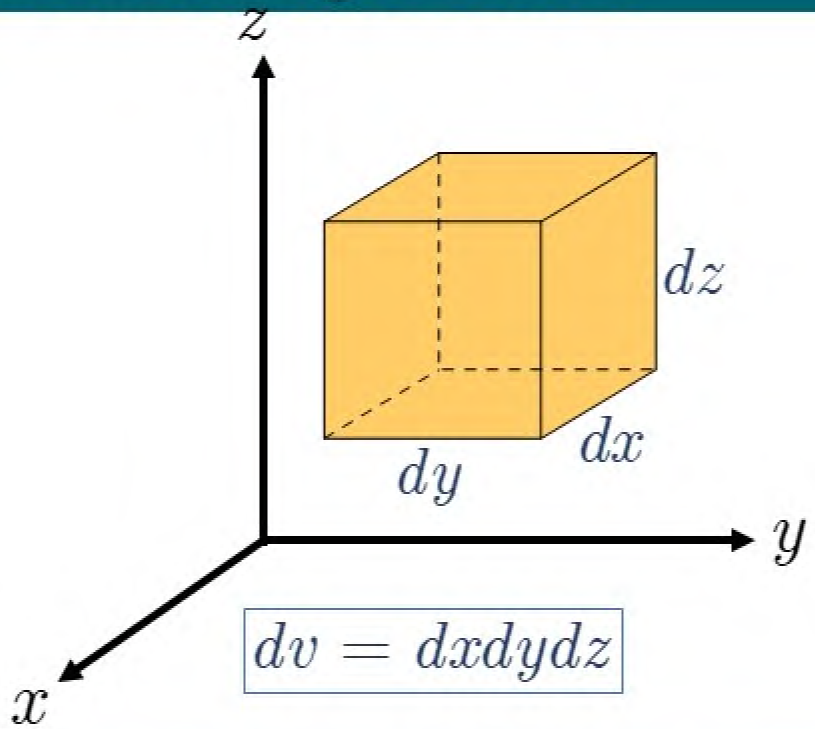
$$\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_z = 1$$
$$\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_z = 0$$

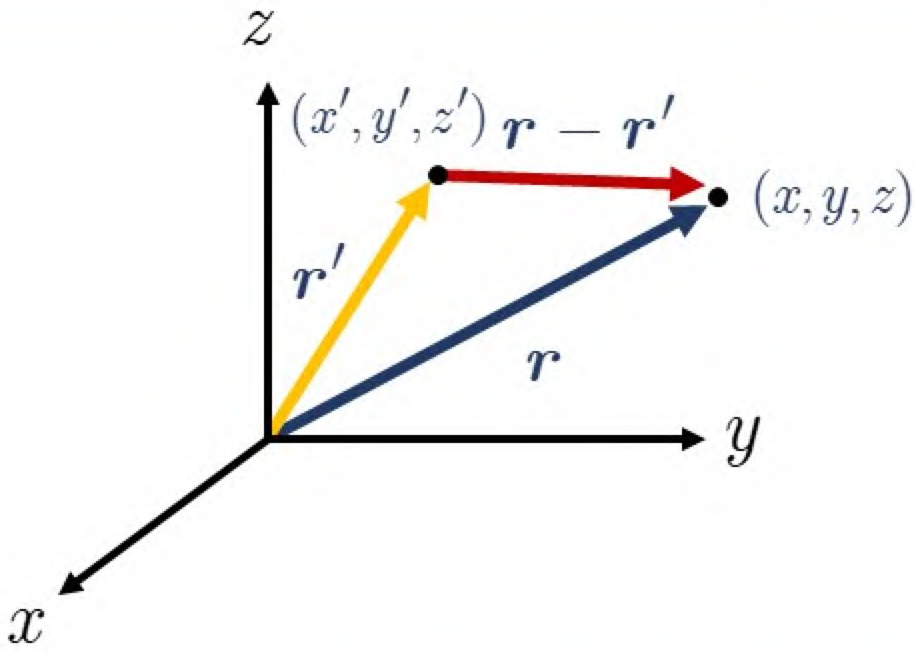




$$\hat{\boldsymbol{e}}_x\times\hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z$$
$$\hat{\boldsymbol{e}}_y\times\hat{\boldsymbol{e}}_x = -\hat{\boldsymbol{e}}_z$$







$$\boxed{dv = dxdydz}$$

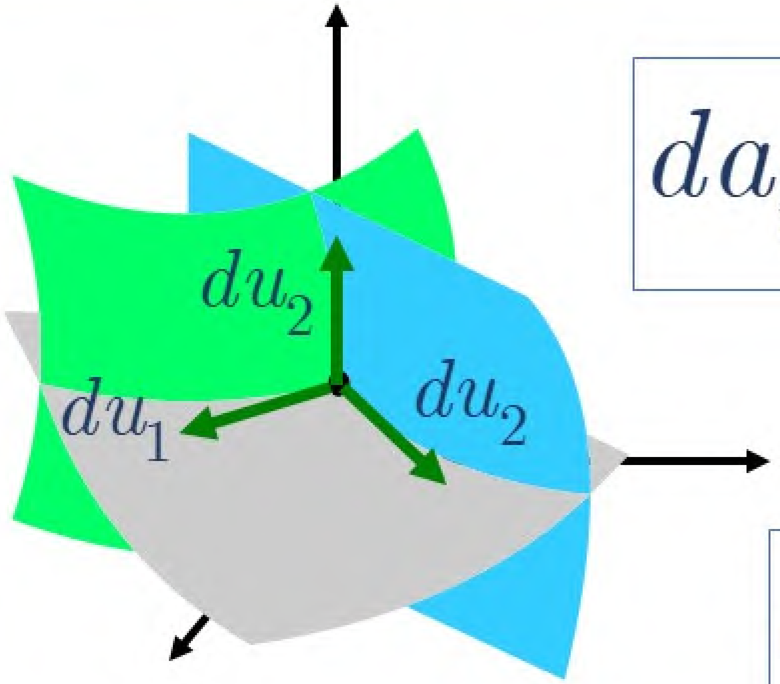
$$\boxed{d\boldsymbol{l} \equiv d\boldsymbol{r} = dx\hat{\boldsymbol{e}}_x + dy\hat{\boldsymbol{e}}_y + dz\hat{\boldsymbol{e}}_z}$$

$$\boxed{\begin{aligned} da_x &= dydz \\ da_y &= dxdz \\ da_z &= dxdy \end{aligned}}$$

$$\boxed{\boldsymbol{r} - \boldsymbol{r}' = (x - x')\hat{\boldsymbol{e}}_x + (y - y')\hat{\boldsymbol{e}}_y + (z - z')\hat{\boldsymbol{e}}_z}$$

فاصله‌ی دو نقطه:

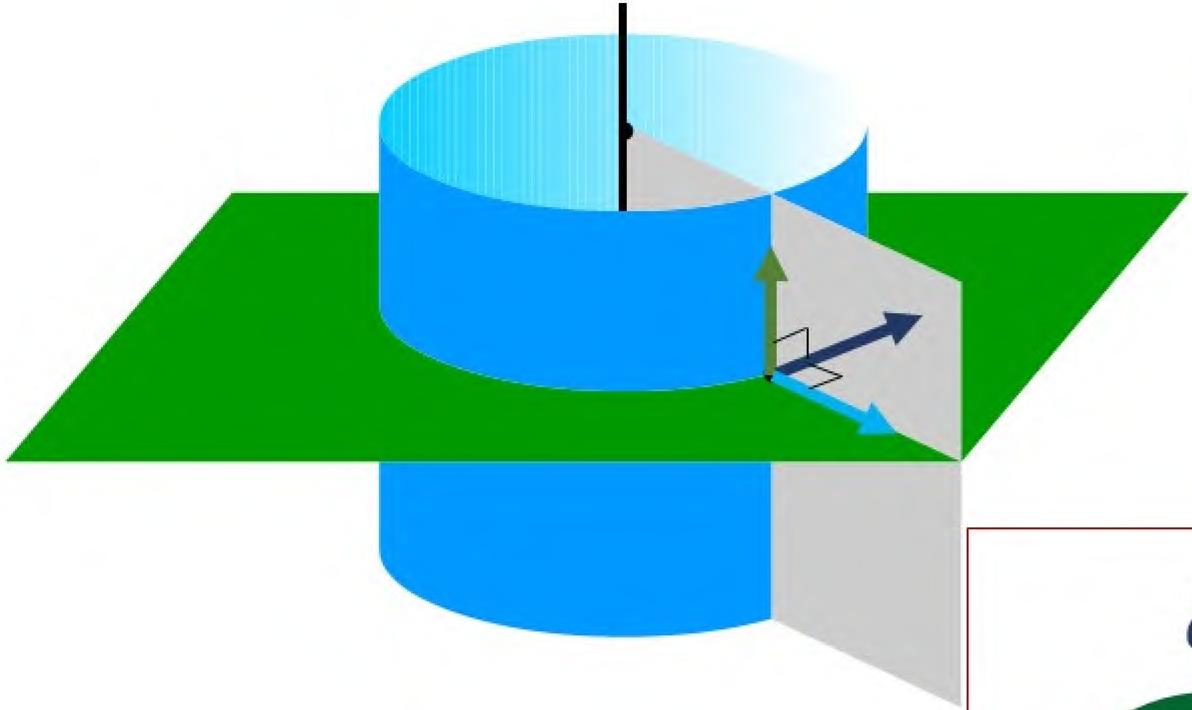
$$\boxed{d = \left|\boldsymbol{r} - \boldsymbol{r}'\right| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$



$$\boxed{da_i = h_j h_k du_j du_k \qquad (i \neq j \neq k)}$$

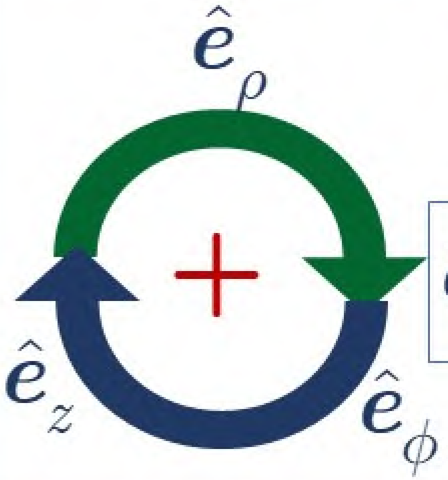
$$\boxed{dv = h_1 h_2 h_3 du_1 du_2 du_3}$$

$$\boxed{d\boldsymbol{l} \equiv d\boldsymbol{r} = h_1 du_1 \hat{\boldsymbol{e}}_{u_1} + h_2 du_2 \hat{\boldsymbol{e}}_{u_2} + h_3 du_3 \hat{\boldsymbol{e}}_{u_3}}$$



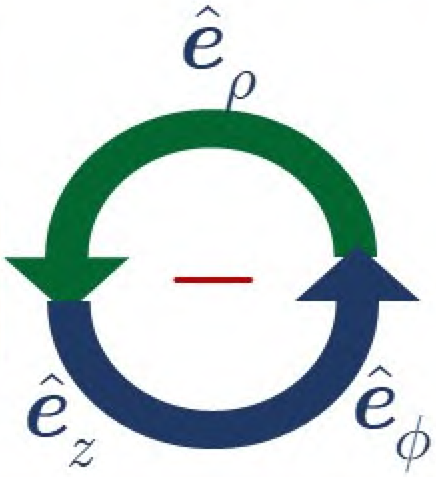
$$\hat{e}_\rho \cdot \hat{e}_\rho = \hat{e}_\phi \cdot \hat{e}_\phi = \hat{e}_z \cdot \hat{e}_z = 1$$

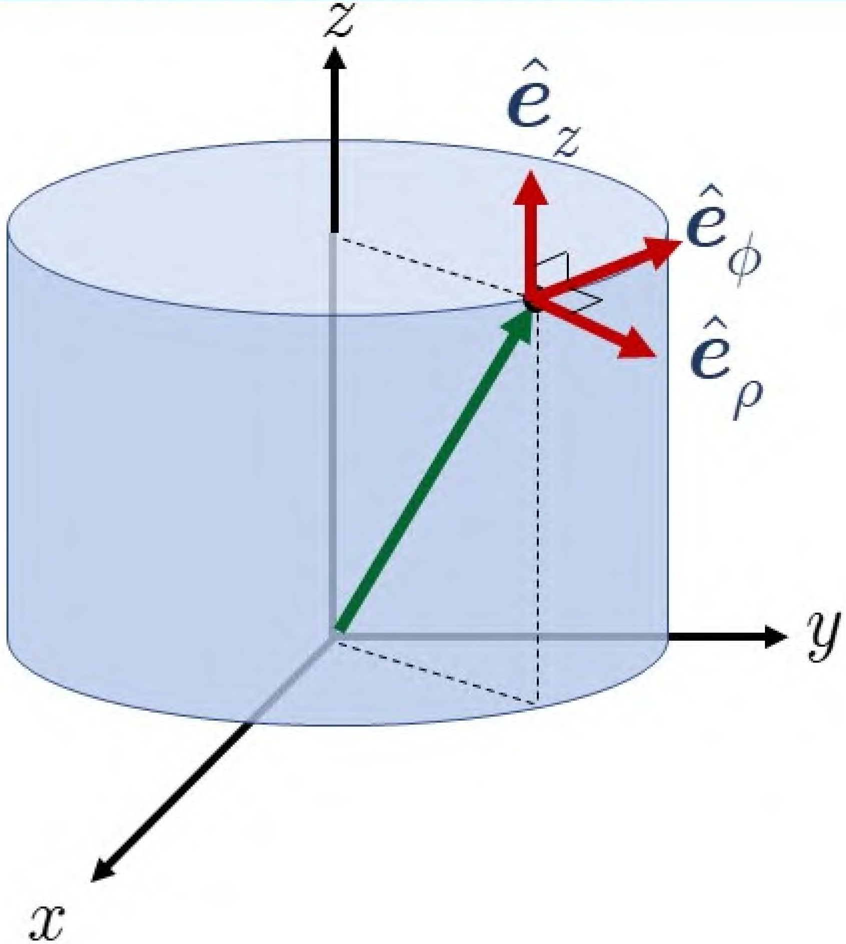
$$\hat{e}_\rho \cdot \hat{e}_z = \hat{e}_\rho \cdot \hat{e}_\phi = \hat{e}_z \cdot \hat{e}_\phi = 0$$



$$\boxed{\hat{e}_\rho \times \hat{e}_\phi = \hat{e}_z}$$

$$\boxed{\hat{e}_\rho \times \hat{e}_z = -\hat{e}_\phi}$$





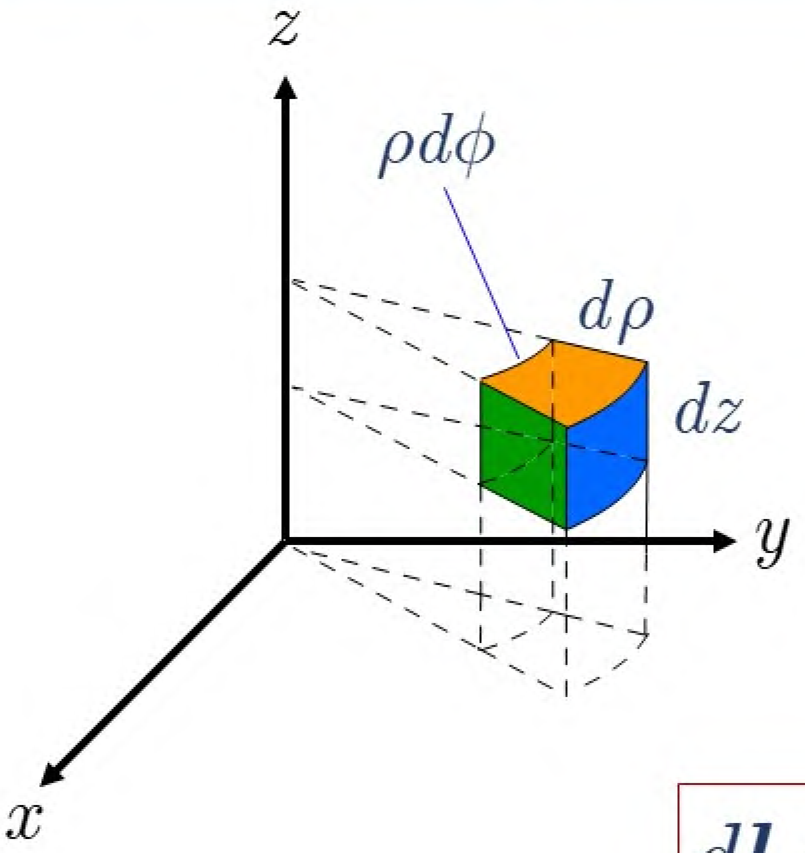
$$\boldsymbol{r} = \rho\hat{\boldsymbol{e}}_\rho + z\hat{\boldsymbol{e}}_z$$

$$|\boldsymbol{r}| = r = \sqrt{\rho^2 + z^2}$$

$$\boldsymbol{A} = A_\rho\hat{\boldsymbol{e}}_\rho + A_\phi\hat{\boldsymbol{e}}_\phi + A_z\hat{\boldsymbol{e}}_z$$

$$|\boldsymbol{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$$

دانشگاه خوارزمی



$$\begin{aligned} h_\rho &= 1 \\ h_\phi &= \rho \\ h_z &= 1 \end{aligned}$$

$$\begin{aligned} da_\rho &= \rho d\phi dz \\ da_\phi &= d\rho dz \\ da_z &= \rho d\rho d\phi \end{aligned}$$

$$dv = \rho d\rho d\phi dz$$

$$d\boldsymbol{l} \equiv d\boldsymbol{r} = d\rho \hat{\boldsymbol{e}}_\rho + \rho d\phi \hat{\boldsymbol{e}}_\phi + dz \hat{\boldsymbol{e}}_z$$

$$d = \left|\boldsymbol{r} - \boldsymbol{r}'\right| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z - z')^2}$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z$$

$$\hat{e}_x = \cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi$$
$$\hat{e}_y = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi$$
$$\hat{e}_z = \hat{e}_z$$

$$\hat{e}_\rho = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y$$
$$\hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y$$
$$\hat{e}_z = \hat{e}_z$$

$$\boldsymbol{A} = A_\rho \hat{\boldsymbol{e}}_\rho + A_\phi \hat{\boldsymbol{e}}_\phi + A_z \hat{\boldsymbol{e}}_z$$

$$\boxed{\begin{aligned}
A_x &= \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_x = A_\rho \hat{\boldsymbol{e}}_\rho\cdot\hat{\boldsymbol{e}}_x + A_\phi \hat{\boldsymbol{e}}_\phi\cdot\hat{\boldsymbol{e}}_x + A_z \hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_x \\
A_y &= \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_y = A_\rho \hat{\boldsymbol{e}}_\rho\cdot\hat{\boldsymbol{e}}_y + A_\phi \hat{\boldsymbol{e}}_\phi\cdot\hat{\boldsymbol{e}}_y + A_z \hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_y \\
A_z &= \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_z = A_\rho \hat{\boldsymbol{e}}_\rho\cdot\hat{\boldsymbol{e}}_z + A_\phi \hat{\boldsymbol{e}}_\phi\cdot\hat{\boldsymbol{e}}_z + A_z \hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_z
\end{aligned}}$$

$$\boxed{\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}}$$

دانشگاه خوارزمی

بردار $A = \hat{e}_x + x\hat{e}_y + (x+y)\hat{e}_z$ در دستگاه مختصات کارتزین داده شده است. مؤلفه‌های این بردار را در دستگاه مختصات استوانه‌ای در نقطه‌ی (-1,2,3) بیابید.

$$\rho = \sqrt{x^2+y^2} = \sqrt{(-1)^2+2^2} = 2.24$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{-1} = 116.57^\circ$$

$$z = 3$$

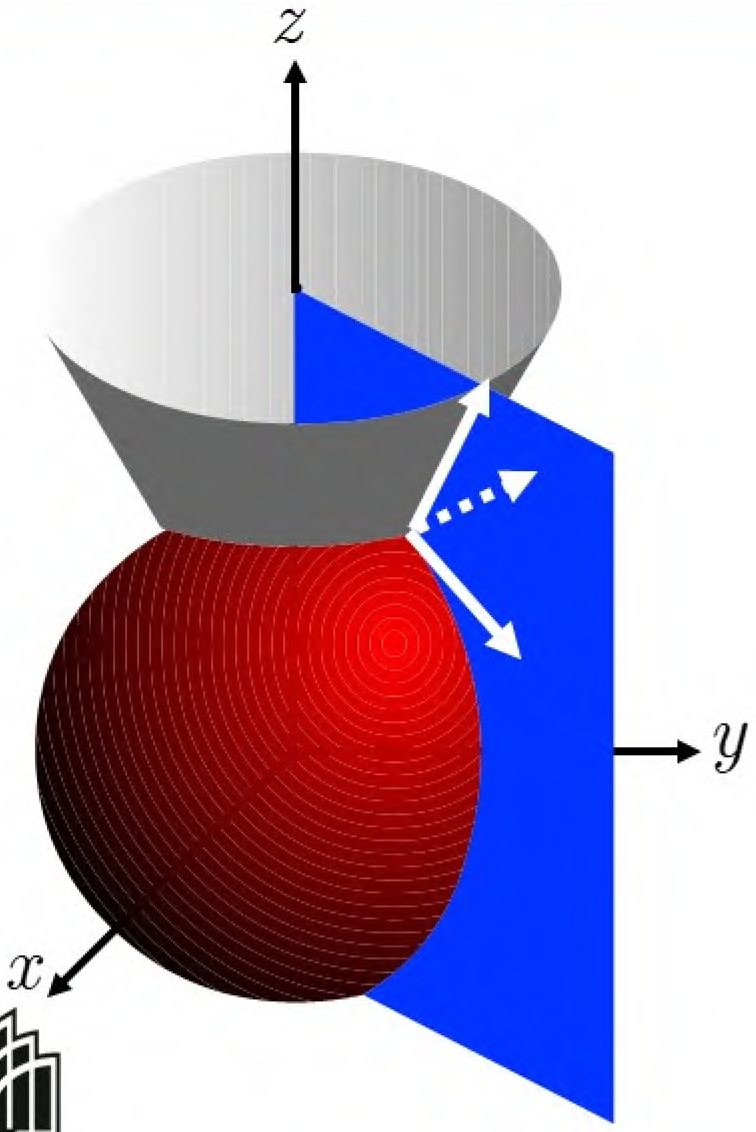
$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x+y \end{pmatrix}$$

$$\begin{aligned} A_\rho &= \cos\phi + x\sin\phi \\ A_\phi &= -\sin\phi + x\cos\phi \\ A_z &= x+y \end{aligned}$$

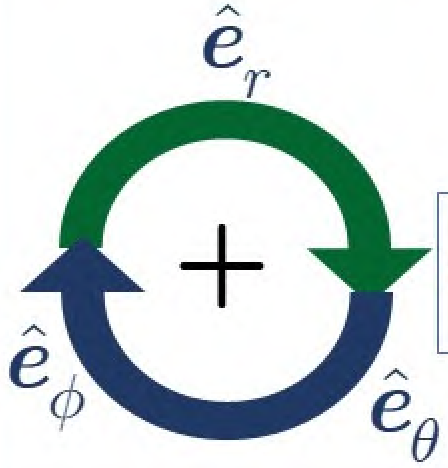
$$\begin{aligned} A_\rho &= \cos\phi + \rho\cos\phi\sin\phi = -1.34 \\ A_\phi &= -\sin\phi + \rho\cos\phi\cos\phi = -0.45 \\ A_z &= \rho\cos\phi + \rho\sin\phi = 1 \end{aligned}$$

بنابر این در نقطه ی (-1,2,3) بردار فوق برابر است با

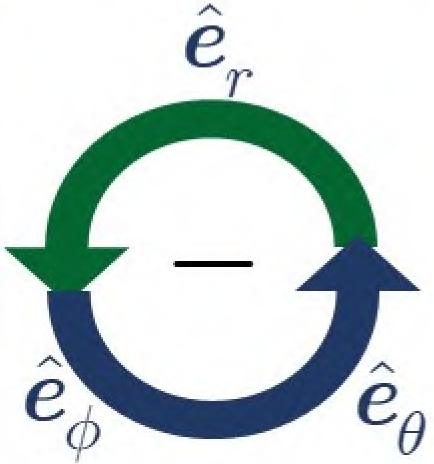
$$\left(A_\rho, A_\phi, A_z\right) = \left(-1.34, -0.45, 1\right)$$

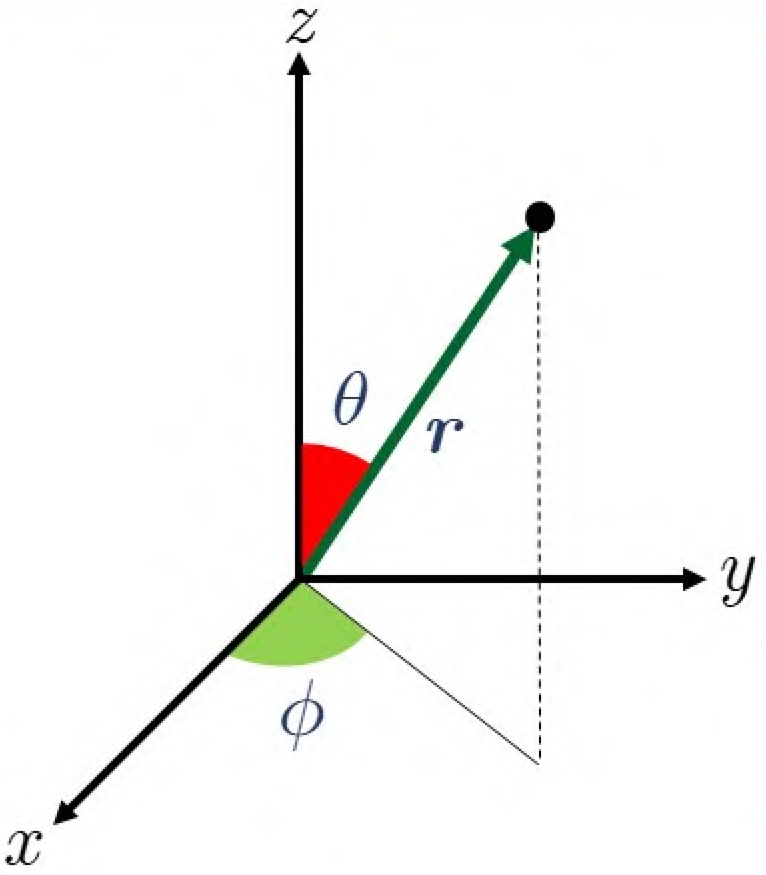


$$\hat{e}_r \cdot \hat{e}_r = \hat{e}_\theta \cdot \hat{e}_\theta = \hat{e}_\phi \cdot \hat{e}_\phi = 1$$
$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_r \cdot \hat{e}_\phi = \hat{e}_\theta \cdot \hat{e}_\phi = 0$$



$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi$$
$$\hat{e}_r \times \hat{e}_\phi = -\hat{e}_\theta$$





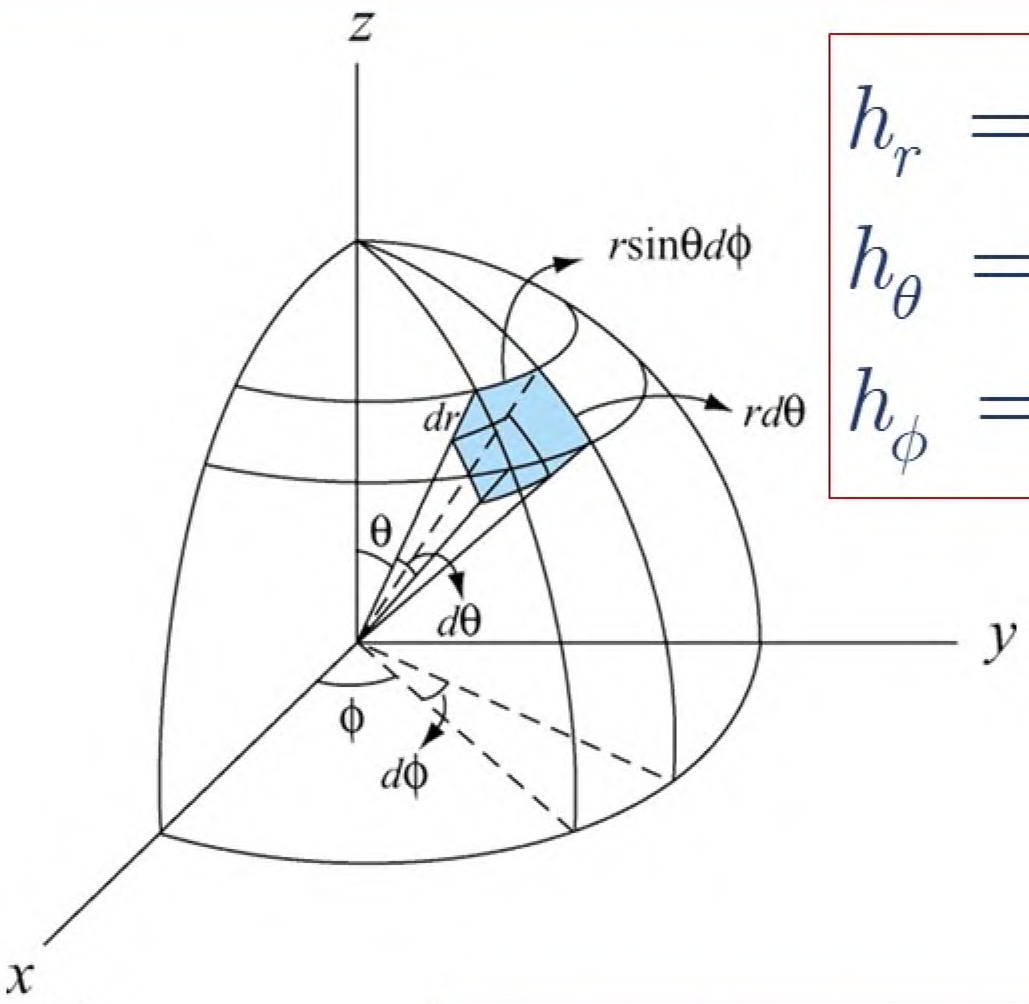
$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_r$$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \phi &= \tan^{-1}\frac{y}{x} \end{aligned}}$$

$$\boxed{d = \left|\boldsymbol{r} - \boldsymbol{r}'\right| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta\cos\theta' - 2rr'\sin\theta\sin\theta'\cos(\phi-\phi')}}$$

دانشگاه خوارزمی



$$h_r = 1$$
$$h_\theta = r$$
$$h_\phi = r\sin\theta$$

$$dv = r^2 \sin\theta dr d\theta d\phi$$

$$da_r = r^2 \sin\theta d\theta d\phi$$
$$da_\theta = r\sin\theta dr d\phi$$
$$da_\phi = r d\theta dr$$

$$d\boldsymbol{l} \equiv d\boldsymbol{r} = dr\hat{\boldsymbol{e}}_r + rd\theta\hat{\boldsymbol{e}}_\theta + r\sin\theta d\phi\hat{\boldsymbol{e}}_\phi$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

شاد و مهربان باشید

دانشگاه خوارزمی